LECTURE 32 ANTIDERIVATIVES

Given that you know the slope of a function $f'(x)$ at every $x \in D$, can you recover the original function $f(x)$? If not, whatelse do you need to pinpoint $f(x)$?

Definition. A function F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Remark. F is not unique for each f.

Example. Given $f(x) = 2x$, $F_1(x) = x^2$ and $F_2(x) = x^2 + 1$ are both antiderivatives of f. In fact, $F(x) = x^2 + C$ for any constant C is an antiderivative. This prompts the following theorem.

Theorem. *If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is*

$$F(x) + C$$

where C is an arbitrary constant.

Example 1. Find an antiderivative $F(x)$ of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

Solution. The general antiderivative is

$$F(x) = x^3 + C.$$

Using the condition $F(1) = -1$, we have

$$-1 = F(1) = (1)^3 + C \implies C = -2.$$

Thus,

$$F(x) = x^3 - 2.$$

Game of recall:

Function	General antiderivative
x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$
$\sin(kx)$	$-\frac{1}{k}\cos(kx) + C$
$\cos(kx)$	$\frac{1}{k}\sin(kx) + C$
$\sec^2(kx)$	$\frac{1}{k}\tan(kx) + C$
$\csc^2(kx)$	$-\frac{1}{k}\cot(kx) + C$
$\sec(kx)\tan(kx)$	$\frac{1}{k}\sec(kx) + C$
$\csc(kx)\cot(kx)$	$-\frac{1}{k}\csc(kx) + C$
e^{kx}	$\frac{1}{k}e^{kx} + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C, \quad x \neq 0$
$\frac{1}{\sqrt{1-k^2x^2}}$	$\frac{1}{k}\sin^{-1}(kx) + C$
$\frac{1}{1+k^2x^2}$	$\frac{1}{k}\tan^{-1}(kx) + C$
$\frac{1}{x\sqrt{k^2x^2-1}}$	$\sec^{-1}(kx) + C$
a^{kx}	$\frac{1}{k\ln(a)}a^{kx} + C, \quad a > 0, a \neq 1$

Antidifferentiation is a linear process, i.e. the antiderivative of $kf(x) \pm hg(x)$ with constants k, h is

$$kF(x) \pm hG(x)$$

where F and G are the antiderivatives of f and g respectively.

Example. Find the general antiderivative of the following functions:

(1) $f(x) = x^5$.
(2) $g(x) = \frac{1}{\sqrt{x}}$.
(3) $h(x) = \sin(2x)$.
(4) $i(x) = \cos\left(\frac{x}{2}\right)$.
(5) $j(x) = e^{-3x}$.

(6) $k(x) = 2^x$.

Definition. The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)\, dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the intergral, and x is the **variable of integration**.

Example. Evaluate

$$\int \left(x^2 - 2x + 5\right) dx$$

Solution.

$$\int \left(x^2 - 2x + 5\right) dx = \frac{x^3}{3} - x^2 + 5x + C$$